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The dependence of the natural frequencies of a one-dimensional elastic system on its length☆

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Abstract

The dependence of the natural frequencies and modes of the oscillations of distributed elastic system with characteristics of the stiffness and density that are variable along a coordinate of the cross section for arbitrary boundary conditions is investigated. It is proved that the presence of an external elastic medium, described by the Winkler model, may lead to an increase in the natural frequencies of the lower oscillation modes when the length of a one-dimensional elastic system is increased. The fine properties of the change in the natural frequencies as a function of the length of the system and the number of the oscillation mode are also established. A numerical-analytical investigation of examples which illustrate the characteristic anomalous behaviour of the lowest natural frequencies is presented.

1. Formulation of the problem

We will consider the problem of determining the natural frequencies ω and modes $u(x)$ of the oscillations of a distributed elastic system, described by a boundary-value problem for a differential equation of the form[1–6]

$$(p(x)u')' + [\lambda r(x) - q(x)]u = 0, \quad 0 < x < l, \quad \lambda = \omega^2 \tag{1.1}$$

for standard sign-definite and smoothness conditions of the functions p, r and q.[2–4] These have a definite physical meaning for distributed elastic systems, namely, p is the stiffness, r is the density per unit length and q is the coefficient of elasticity of the external medium (the Winkler model). At the left end $x = 0$ and right end $x = 1$ of the interval the following boundary conditions of elastic fixing of the system are specified

$$\begin{aligned} &\alpha_0 p(0)u'(0) - \beta_0 u(0) = 0, \quad \alpha_l p(l)u'(l) + \beta_l u(l) = 0 \\ &\alpha_{0,\,l}, \beta_{0,\,l} \geq 0, \quad \alpha_{0,\,l} + \beta_{0,\,l} > 0 \end{aligned} \tag{1.2}$$

The coefficients $\alpha_{0,\,l}$ and $\beta_{0,\,l}$ represent the relative effect of the distributed and concentrated elasticity. In particular, when $\alpha_0 = \alpha_l = 0$ we have the boundary conditions of rigid fixing: $u(0) = u(l) = 0$; when $\beta_0 = \beta_l = 0$ the ends of the system are free: $u'(0) = u'(l) = 0$. Conditions of these forms may be satisfied at one or both ends of the interval; the length of the interval is assumed to be limited: $l < \infty$.

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The problem of determining and analysing the natural frequencies and forms (1.1) and (1.2), i.e. the Sturm-Liouville problem of the natural values and functions and its generalizations has been comprehensively investigated in connection with applications to the theory of elasticity, hydrodynamics and acoustics. A number of analytical and numerical methods of estimation and approximate solution of the problem have been developed. The fundamental properties of the problem have been established: the existence of a denumerable discrete set of natural frequencies $\omega_n = \sqrt{\lambda_n}$ and forms $u_n(x) = u(x, \lambda_n)$ $(n = 0, 1, 2, \ldots)$, the oscillation properties, the asymptotic behaviour as $n \to \infty$, etc.

However, the effect of the length of the interval in which a change in the argument x occurs, i.e. the values of the parameter $l > 0$ has not been investigated in known publications. In theoretical and applied formulations of the problems it is usually assumed that $l = l_0 = \text{const}$ and, by means of normalization, the length is reduced to the value $l_0 = 1$. An analysis of the solution based on Sturm's comparison theorems[2–4] and using an accelerated convergence numerical-analytical method[1,5] indicates a significant influence of the parameter l on the natural frequencies of the system. It is convenient to use this fundamental property in approximate procedures for calculating the natural frequencies $\omega_n(l)$ and the natural forms $u_n^*(x, l) = u_n \|u_n\|_r^{-1}$, orthonormalized with weight $r(x)$, where the square of the norm $\|u_n\|_r^2 = (u_n, u_n)_r$ is the weighted scalar product.

In the first place, a local investigation of the relation $\lambda_n(l)$ in the neighbourhood of the fixed value $l = l_0$, i.e. for $l = l_0 + \delta l$, where the variation of the length $|\delta l| \ll l_0$, is of theoretical and applied interest. After normalizing the argument x by l_0 we can consider the interval $0 \le x \le 1 + a$, where $a = \delta l / l_0$, $|a| \ll 1$, see Sections 3 and 4. Then, using a numerical-analytical procedure of continuation with respect to the parameter a and the accelerated convergence method, we can investigate the relation $\lambda_n(l)$ for variations $\delta l \sim l_0$, i.e. $a \sim 1$. To fix our ideas, the quantity l_0 will be chosen as the minimum value of l and the investigation will be carried out for $a > 0$. In general, this assumption is unnecessary.

We will obtain expressions for the "sensitivity coefficients", which are defined by the first and second derivatives $\lambda_n'(l_0)$, $\lambda_n''(l_0)$ with respect to l for $l = l_0$. They represent the local relation $\lambda_n(l)$ in the neighbourhood of the value $l = l_0$. This corresponds, in the normalized version, to the expressions $\lambda_n'(1) = \lambda_n'$, $\lambda_n''(1) = \lambda_n''$, i.e. the derivatives with respect to the parameter a for $a = 0$. The expressions for subsequent coefficients, defined by higher-order derivatives, are determined similarly.

2. Determination of the local dependence of the natural oscillation frequencies on the length parameter of the system

The required local characteristics of the natural oscillation frequencies are constructed using the well-known solution of the self-adjoint boundary-value problem for natural frequencies and the functions (1.1) and (1.2) for $l = l_0$ ($l = 1$, i.e. $a = 0$, when the argument x is normalized by l_0). It is natural to determine the approximate natural frequencies ω_n and eigenvalues λ_n in the form

$$\begin{aligned}
\omega_n(l) &= \omega_n(l_0) + \omega_n'(l_0)\delta l + \frac{1}{2}\omega_n''(l_0)\delta l^2 + O(\delta l^3) \\
\lambda_n(l) &= \lambda_n(l_0) + \lambda_n'(l_0)\delta l + \frac{1}{2}\lambda_n''(l_0)\delta l^2 + O(\delta l^3) \\
\omega_n &= \sqrt{\lambda_n}, \quad \omega_n' = \frac{1}{2}\frac{\lambda_n'}{\sqrt{\lambda_n}}, \quad \omega_n'' = \frac{\lambda_n''}{\sqrt{\lambda_n}} - \frac{1}{4}\frac{\lambda_n'^2}{\lambda_n^{3/2}}
\end{aligned} \tag{2.1}$$

when $|\delta l|$ is fairly small. Representations similar to (2.1) can be written for the natural forms of oscillations u_n

$$\begin{aligned}
u_n(x, l) &= u_n(x, l_0) + \upsilon_n(x)\delta l + \frac{1}{2}w_n(x)\delta l^2 + O(\delta l^3) \\
\upsilon_n &= \frac{\partial u_n(x, l_0)}{\partial l}, \quad w_n = \frac{\partial^2 u_n(x, l_0)}{\partial l^2}
\end{aligned} \tag{2.2}$$

Thus, we assume that the solution λ_n, $u_n(x, l)$ of problem (1.1), (1.2) is known for the particular value $l = l_0$. The derivatives λ_n', λ_n'' and the sensitivity functions υn and w_n are found by differentiating with respect to l. In particular, to find $\lambda'_n(l_0)$ and $v_n(x)$ we use the relations

$$(pv')' + (\lambda r - q)v = -\lambda' ru, \quad \alpha_0 pv' - \beta_0 v = 0, \quad x = 0$$
$$\alpha_l((\lambda r - q)u + pv') + \beta_l(u' + v) = 0, \quad x = l_0 \tag{2.3}$$

For brevity we have not written the arguments and subscripts in relations (2.3).

We multiply the equations for u (1.1) by v and the equation for v (2.3) by u and subtract the second result from the first. Integrating the difference with respect to x in the section $0 \le x \le l_0$, taking boundary conditions (1.2) and (2.3) into account, we obtain the required expression for λ'_n

$$\lambda'_n(l_0) = -[pu'^2_n + (\lambda_n(l_0)r - q)u_n^2]\|u_n\|_r^{-2}, \quad x = l_0 \tag{2.4}$$

These representations are independent of the form of the functions $v_n(x)$. From expressions (2.4) we can determine particular expressions in the case of boundary conditions corresponding to rigid fixing ($\alpha_l = 0$) and a free ($\beta_l = 0$) right end $x = l_0$; by conditions (1.2) we obtain, respectively,

$$\lambda'_n(l_0) = -p(l_0)u'^2_n(l_0, l_0)\|u_n\|_r^{-2}, \quad n = 1, 2, \ldots, \quad u(l_0, l_0) = 0 \text{ when } \alpha_l = 0 \tag{2.5}$$

$$\lambda'_n(l_0) = -[\lambda_n(l_0)r(l_0) - q(l_0)]u_n^2(l_0, l_0)\|u_n\|_r^{-2}, \quad n = 0, 1, 2, \ldots$$
$$u'(l_0, l_0) = 0 \text{ when } \beta_l = 0 \tag{2.6}$$

The quantities $\omega'_n(l_0)$ are calculated from formulae (2.1).

For the boundary condition of rigid fixing, according to relations (2.5) the quantities λ'_n are negative for all $n = 1, 2, \ldots$, i.e. all the eigenvalues λ_n and the natural frequencies ω_n decrease as the interval is extended by moving the right end rightwards. This property corresponds to physical considerations and is known in the literature.[1–4] When $q(l_0) = 0$ or for sufficiently small $q > 0$, the values of ω'_n, λ'_n become negative for general conditions of elastic fixing (2.4) and for a free right end (2.6) of the system, which also corresponds to the effect of decreasing all the quantities ω_n, $\lambda_n (n = 0, 1, \ldots)$ for sufficiently small $|\delta l| > 0$.

If the action of the external elastic medium, characterized by the coefficient $q(x)$, is "fairly large" locally in the neighbourhood of the point $x = l_0$, and is "relatively small" integrally in the section $0 \le x \le l_0$, the nature of the sign-definiteness of the quantities $\omega'_n(l_0)$, $\lambda'_n(l_0)$ (2.6) becomes more complicated. The following assertion regarding the local behaviour of the eigenvalues $\lambda_n(l)$ in the neighbourhood of $l = l_0$ holds.

Theorem. *The following inequalities hold for the oscillation frequencies when $l = l_0$*

$$\omega'_0 > 0, \quad \omega'_1 > 0, \ldots, \omega'_{k-1} > 0, \quad \omega'_k \ge 0$$
$$\lambda'_0 > 0, \quad \lambda'_1 > 0, \ldots, \lambda'_{k-1} > 0, \quad \lambda'_k \ge 0 \tag{2.7}$$
$$\omega'_n < 0, \quad \lambda'_n < 0, \quad n \ge k + 1$$

This corresponds to a local increase in the natural frequencies and the forms of the lower modes $n = 0, 1, \ldots, k - 1$ and a non-decreasing k-th mode (in the linear approximation) as $l \le l_0$ increases; the subsequent natural frequencies and eigenvalues decrease as l increases.

The sufficient constructive conditions can be established using Rayleigh's principle and the Rayleigh-Ritz method,[1–3] which enable one to obtain effective upper estimates $\omega^*_n(l_0)$ and $\lambda^*_n(l_0)$ of the required frequencies $\omega_n(l_0)$ and eigenvalues $\lambda_n(l_0)$. In particular, for case (2.6) when $n = 0$ we have

$$\omega^*_0(l_0) = \sqrt{\lambda^*_0(l_0)}$$
$$\lambda^*_0(l_0) = \int_0^{l_0}[p(x)\psi'^2_0(x) + q(x)\psi_0^2(x)]dx\left[\int_0^{l_0} r(x)\psi_0^2(x)dx\right]^{-1} \tag{2.8}$$

where $\psi_0(x)$ is the test function and $\psi'_0(l_0) = 0$. To calculate the next λ^*_n it is necessary to construct a system of test functions $\{\psi_n(x)\}$. Hence, if inequalities (2.7) occur when the values of λ^*_n are substituted into expressions (2.6), they necessarily hold for the exact quantities $\lambda_n \leq \lambda_n^*$, $n \leq k$. The set of functions $p(x)$, $r(x)$ and $q(x)$, which satisfy conditions (2.7), is non-empty (see Sections 3 and 4).

The more detailed property of the natural frequencies $\omega_n(l)$ and eigenvalues $\lambda_n(l)$ established in the theorem has not previously been pointed out. Moreover, there are doubtful assertions (see Ref. 4, p. 512), which contradict the assertions of the theorem and the calculations of specific examples (see below in Sections 3 and 4): thus, it is stated that extension of the interval implies a reduction in all the eigenvalues for the Sturm-Liouville problem with boundary conditions of the second kind.

Note that the rough property (ignoring the function $q(x)$) of the natural frequencies $\omega_n(l) \sim l_n^{-1}$ and the values $\lambda_n(l) \sim l^{-2}$, that they decrease with l when $n \gg 1$, obviously follows from the asymptotic estimates and has been investigated in detail in the classical books[2,3] and subsequent publications.[1,6]

In the critical case $\lambda_n'(l_0) \approx 0$ the effect of an increase or decrease in the values of $\lambda_n(l)$ in the neighbourhood of $l = l_0$ is determined by the value and sign of the second derivative $\lambda_n''(l_0)$ according to formula (2.1). Expressions for λ_n'' can be obtained using relations (2.3) and similarly for the unknown w

$$\begin{gathered}(pw')' + (\lambda r - q)w = -\lambda'' ru - 2\lambda' r v; \quad \alpha_0 pw' - \beta_0 w = 0, \quad x = 0\\ \alpha_l(((\lambda r - q)u)' - 2p'v' + 2pv'' + pw)' + \beta_l(u'' + 2v' + w) = 0, \quad x = l_0\end{gathered} \tag{2.9}$$

(the arguments and subscripts are omitted here for brevity). In exactly the same way as when calculating λ_n (see above), the procedure of multiplying Eq. (2.9) for u_n by w_m and Eq. (2.9) for w_n by u_n, subtracting and integrating the difference, leads to extremely cumbersome formulae

$$\begin{gathered}\lambda_n''(l_0) = -2\lambda_n'(l_0)(v_n, u_n)_r \|u_n\|_r^{-2} + [(\lambda_n r' - q' + 2\lambda_n' r)u_n^2 -\\ - p'u_n'^2 + 2(\lambda_n r - q)u_n v_n + pu_n' v_n')]_{x = l_0} \|u_n\|_r^{-2}\end{gathered} \tag{2.10}$$

The right-hand side of Eq. (2.10) is independent of w_n. In formulae (2.10) it is assumed that the functions p, r and q are differentiable at the point $x = l_0$. When $\lambda_n' = 0$ the corresponding expression for λ_n'' is simplified considerably. The required quantities λ_n, λ_n', λ_n'' can be calculated in practice by numerical-analytical methods. According to Eq. (2.10) it is required to determine the functions $v_n(x)$, $v_n'(x)$ for all $0 \leq x \leq l_0$ by solving inhomogeneous boundary-value problems (2.3). By virtue of the homogeneity of expressions (1.1), (1.2), (2.3) and (2.9) with respect to u and v we can confine ourselves to finding v_n and v_n' by solving the Cauchy problem with the conditions

$$v_n(0) = \frac{\alpha_0 p(0)}{\alpha_0 p(0) + \beta_0}, \quad v_n'(0) = \frac{\beta_0}{\alpha_0 p(0) + \beta_0}, \quad n = 0, 1, \ldots$$

In practical calculations it is preferable to use the procedure of continuation with respect to the parameter l based on the accelerated convergence method.[1,5] The results discussed indicate the presence of previously unknown qualitative features of the behaviour of the natural frequencies of oscillations of distributed elastic systems.

3. Test examples

We will calculate some model examples which illustrate the effect of the increase in the lowest eigenvalues and natural frequencies as the length of the system l increases. For convenience we will assume $l = 1 + a$, $\lambda = \lambda(a)$ and $u = u(x, a)$; the coefficients of Eq. (1.1) and the boundary conditions (1.2) are taken in the form

$$p(x) \equiv 1, \quad r(x) = (2 + x)^{-1}, \quad q = x^2, \quad u'(0) = u'(1 + a) = 0 \tag{3.1}$$

The minimum (zero) eigenvalue $\lambda_0(0)$ for $a = 0$, i.e. when $l = l_0 = 1$, is equal to $\lambda_0(0) = 0.796$; the next eigenvalue $\lambda_1(0) = 25.505$. They are found by using the accelerated convergence method.[1,5] We will calculate the coefficient in the square brackets for λ_0 from relation (2.6); it is equal to $\lambda_0 r(1) - q(1) = -0.73474 < 0$. The fact that it is negative indicates the increase (or decrease) in the value of $\lambda_0(a)$, i.e. $\lambda_0(a) \gtrless \lambda_0(0)$, when the parameter $a \gtrless 0$ increases (or decreases), see Fig. 1. The eigenvalues λ_n, $n \geq 1$ decrease as the value of a increases and increase when the value of a decreases. In particular, for λ_1 we have $\lambda_1 r(1) - q(1) = 7.501 > 0$, see Fig. 1. When $q = 9x^2$ we have $\lambda_0(0) = 5.625$;

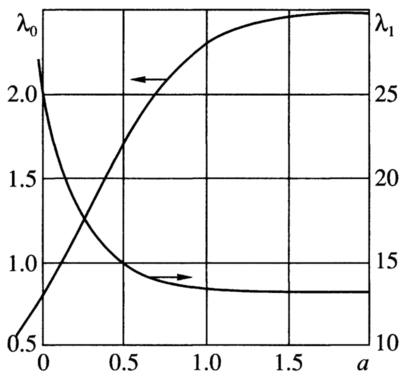

Fig. 1.

the corresponding curve of $\lambda_0(a)$ is similar to the curve of $\lambda_0(a)$ in Fig. 1, but now $\lambda_0''(1) < 0$, i.e. the curve is convex upwards, rather than downwards, as in Fig. 1.

We will consider another example which enables us to establish the property, indicated in the theorem, that the subsequent lower eigenvalues λ_n, $n \geq 1$ increase. We will take the class of functions $q(x)$, which define the coefficient of elasticity of the external elastic medium, which increase more sharply than in (3.1) when $x \approx 1$. Suppose the coefficients p, r and q of Eq. (1.1) have the form

$$p(x) = r(x) \equiv 1, \quad q(x) = (q_0 + x)^m, \quad q_0 > 0, \quad m \gg 1 \tag{3.2}$$

By choosing the parameters q_0 and m we can obtain a sharp increase in the coefficient of elasticity $q(x)$ near the right end $x = 1$ for a moderate increase in $\lambda_n(a)$. When $q_0 = 0.4$ and $m = 8$ we obtain $\lambda_0(a) = 2.4346$ and $\lambda_1(0) = 16.897$; the values of the coefficient in the square brackets of (2.6) are negative, since $q(1) = 25.6 > \lambda_{0,1}(0)$. The corresponding curves of $\lambda_0(a)$ and $\lambda_1(a)$ for $-0.3 \leq a \leq 0.3$ are shown in Fig. 2. Like the curve of $\lambda_0(a)$ in Fig. 1, they illustrate the property that the natural frequencies $\omega_{0,1}(a)$ and eigenvalues $\lambda_{0,1}(l)$ for the lower modes $n = 0,1$ increase when the parameter a increases in the neighbourhood of $a = 0$, as established in the theorem.

It is interesting to note, however, that the function $\lambda_1(a)$ is non-monotonic as a decreases ($a < 0$). In the neighbourhood of the value $l \approx 0.9$ ($a \approx -0.1$) a pronounced minimum is observed.

When $a \to -1$, which implies $l \to 0$, we have the asymptotic form $\lambda_1 \sim (\pi/l)^2 \to \infty$, which corresponds to physical considerations. The eigenvalue $\lambda_0(a)$ decreases monotonically as $a \to -1$, i.e. $l \to 0$, and approaches the values $\lambda_0(-1) = q_0^m = 1/256$.

Using this approach one can investigate the case of other values of the parameters q_0 and m, and also more complex expressions for the coefficients p, r and q. The nature of the effect can be illustrated most convincingly using examples of systems with piecewise-constant characteristics. They allow of complete integration and can be reduced to finite transcendental equations for determining and analysing the natural frequencies. Such systems are of certain interest when solving and analysing applied problems.

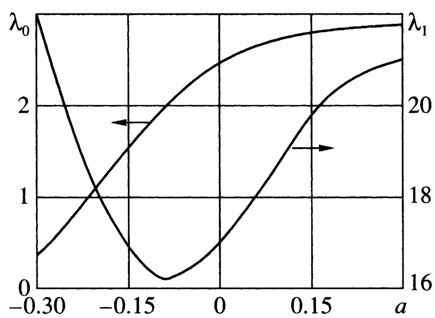

Fig. 2.

4. Systems with piecewise-constant characteristics

Problem (1.1), (1.2) for an elastic system with piecewise-continuous coefficients $p(x)$, $r(x)$ and $q(x)$ requires the development of special calculation algorithms. It can be investigated using a variational approach and the Rayleigh-Ritz method. In practice elastic systems can often be described by an equation of the form (1.1) with piecewise-constant characteristics. This enables one to integrate the equations on intervals where the coefficients are constant and reduce the solution of the problem to an algebraic transcendental equation containing power-law, trigonometric and exponential functions for determining the eigenvalues and natural frequencies of the system. Instead of a discontinuous function u' it is more convenient to use the continuous physical variable $\theta = pu'$, where the coefficient p has a kink at the points of discontinuity. The variable u has a kink at the points of discontinuity of the stiffness coefficient p.

1°. We will consider the relatively simple situation when the elastic system (a string, an elastic shaft or a beam) has constant characteristics p, $r = \text{const} > 0$, while the external elastic medium contains two parts where the stiffness coefficient $q(x)$ is constant. As a result of normalization we obtain a problem of the form

$$u'' + [\lambda - q(x)]u = 0; \quad q \equiv 0, \quad 0 \le x \le 1; \quad q \equiv q_0, \quad 1 < x \le l$$
$$\alpha_{0,l}u' \mp \beta_{0,l}u = 0, \quad x = 0, \; l = 1 + a \tag{4.1}$$

The general solution of Eq. (4.1) when $\lambda \ge q_0$ can be represented in the form ($a_{1,2}$, $b_{1,2} = \text{const}$)

$$u = a_1 \sin\nu x + b_1 \cos\nu x, \quad \nu(a) \equiv \sqrt{\lambda(a)}, \quad 0 \le x \le 1 \tag{4.2}$$

$$u = a_2 \sin\omega(l - x) + b_2 \cos\omega(l - x), \quad \omega = \sqrt{\lambda - q_0}, \quad 1 < x \le l \tag{4.3}$$

The values of $u(x)$ and $u'(x)$ must be identical when $x = 1 \pm 0$.

When $q_0 \ge \lambda$ for $0 \le x \le 1$ expression (4.2) remains true. For $1 < x \le l$ the trigonometric functions in (4.3) must be replaced by hyperbolic functions, which is equivalent to the replacements $\omega \to i\omega$, $ia_2 \to a_2$.

The characteristic equation for determining the eigenvalues λ is obtained from the condition for the fourth-order determinant to be equal to zero.

When $\lambda \ge q_0$ we have

$$\Delta(\lambda) = (-\alpha_0\alpha_l\nu^2 + \beta_0\beta_l)\omega\sin\nu\cos\omega a - (\alpha_0\beta_l\nu^2 - \beta_0\alpha_l\omega^2)\sin\nu\sin\omega a +$$
$$+ (\alpha_0\beta_l + \beta_0\alpha_l)\nu\omega\cos\nu\cos\omega a + (-\alpha_0\alpha_l\omega^2 + \beta_0\beta_l)\nu\cos\nu\sin\omega a = 0 \tag{4.4}$$

($\Delta(\lambda)$ is an odd function of the variables ν and ω). When $q_0 \ge \lambda$ we have a representation of the form (4.4) in which we must make the substitutions

$$\Delta \to i\Delta, \quad \omega \to i\omega, \quad \sin\omega a \to i\,\text{sh}\,\omega a, \quad \cos\omega a \to \text{ch}\,\omega a$$

with the corresponding expression for $\omega(\omega = \sqrt{q_0 - \lambda})$. When $a = 0$ Eq. (4.4) has a standard trigonometric form (after contraction at $\omega \ne 0$)

$$(\alpha_l\beta_0 + \alpha_0\beta_l)\nu\cos\nu - (\alpha_0\alpha_l\nu^2 - \beta_0\beta_l)\sin\nu = 0, \quad \nu_n(0) = \sqrt{\lambda_n(0)} \tag{4.5}$$

The roots $\nu_n(0)$ of Eq. (4.5) can be found numerically for all $n = 0, 1, 2, \ldots$ and specified $\alpha_{0,l}$, $\beta_{0,l}$. The corresponding values of $\nu_n(a)$ when $a \ne 0$ are calculated using an extremely lengthy procedure of continuation with respect to the parameter. Hence, it is of interest at the initial stage to investigate the limiting situations when there are boundary conditions corresponding to fixed or free ends.

2°. Suppose both ends are free, i.e. we have $\beta_{0,l} = 0$, $\alpha_{0,l} = 1$ in problem (4.1). From relations (4.2) and (4.3) we obtain the following representations for the eigenfunctions and characteristic equations for $q_0 \gtrless \lambda$ respectively

$$\begin{aligned} &u = b_1\cos\nu x, \quad 0 \le x \le 1 \\ &u = b_2\operatorname{ch}\omega(l-x), \quad \omega = \sqrt{q_0-\lambda}, \quad \nu\sin\nu\operatorname{ch}\omega a - \omega\cos\nu\operatorname{sh}\omega a = 0 \\ &u = b_2\cos\omega(l-x), \quad \omega = \sqrt{\lambda-q_0}, \quad \nu\sin\nu\cos\omega a + \omega\cos\nu\sin\omega a = 0 \\ &1 < x \le 1+a, \quad \nu_n(0) = n\pi, \quad n = 0, 1, \dots \end{aligned} \tag{4.6}$$

It follows from relations (4.6) that the lower natural frequencies $\omega_n(a) = \sqrt{\lambda_n(a)}$, such that $\lambda_n(0) < q_0$, increase as $a \to \infty$ and approach the roots of the corresponding equation $\nu\sin\nu = \omega\cos\nu$. By a standard method of analysis we can similarly establish that the higher natural frequencies $\omega_n(a) = \sqrt{\lambda_n(a)}$, such that $\lambda_n(0) > q_0$, decrease. In the limiting case, if, for certain $n = n^*$, the quantity $\lambda_{n^*}(1) = q_0 = (n^*\pi)^2$, then $\lambda_{n^*}(l) \equiv (n^*\pi)^2$ irrespective of the value of $a \ge 0$, i.e. $\omega_{n^*}(a) \equiv n^*\pi$. When $n < n^*$ the values of $\lambda_n(a)$ increase as the length parameter a increases and when $n > n^*$ the values of $\lambda_n(a)$ decrease; the behaviour of the natural frequencies $\omega_n(a)$ is similar.

We will briefly consider the case when the left end $(x = 0)$ is fixed – $u(0) = 0$, while the right end $(x = l)$ is free – $u'(l) = 0$. Relations of the type (4.6) become

$$\begin{aligned} &u = a_1\sin\nu x, \quad 0 \le x \le 1 \\ &u = b_2\operatorname{ch}\omega(l-x), \quad \omega = \sqrt{q_0-\lambda}, \quad \nu\cos\nu\operatorname{ch}\omega a + \omega\sin\nu\operatorname{sh}\omega a = 0 \\ &u = b_2\cos\omega(l-x), \quad \omega = \sqrt{\lambda-q_0}, \quad \nu\cos\nu\cos\omega a - \omega\sin\nu\sin\omega a = 0 \\ &1 < x \le l, \quad \nu_n(0) = (n-1/2)\pi, \quad n = 1, 2, \dots \end{aligned} \tag{4.7}$$

As above, the lower eigenvalues $\lambda_n(a)$ such that $\lambda_n(0) < q_0$, increase, while the higher eigenvalues $\lambda_n(0) > q_0$, increase; when $q_0 = (n^* - 1/2\pi)^2$ we have $\lambda_{n^*}(a) \equiv (n^* - 1/2\pi)^2$, i.e. $\omega_{n^*}(a) \equiv (n^* - 1/2)\pi$.

3°. We will investigate the case of a fixed right end: $u(l) = 0$. Suppose the left end is also fixed – $u(0) = 0$; then, like relations (4.6) and (4.7), we will have the following representations for the eigenfunctions and characteristic equations

$$\begin{aligned} &u = a_1\sin\nu x, \quad 0 \le x \le 1 \\ &u = a_2\operatorname{sh}\omega(l-x), \quad \omega = \sqrt{q_0-\lambda}, \quad \omega\sin\nu\operatorname{ch}\omega a + \nu\cos\nu\operatorname{sh}\omega a = 0 \\ &u = a_2\sin\omega(l-x), \quad \omega = \sqrt{\lambda-q_0}, \quad \omega\sin\nu\cos\omega a + \nu\cos\nu\sin\omega a = 0 \\ &1 < x \le 1+a, \quad \nu_n(0) = n\pi, \quad n = 0, 1, \dots \end{aligned} \tag{4.8}$$

According to the representations (4.8), all eigenvalues $\lambda_n(a)$ decrease as the parameter a increases independently of the quantities $\lambda_n(0) \gtrless q_0$. If $q_0 = (n^*\pi)^2$, then $\lambda_{n^*}(a) \equiv q_0$ when $a \ge 0$; the behaviour of $\omega_n(a)$ and $\omega_{n^*}(a)$ is similar, but $u_{n^*} \equiv 0$.

Suppose the left end is free: $u(0) = 0$; we then obtain the following representations for the eigenfunctions and characteristic equations

$$\begin{aligned} &u = b_1\cos\nu x, \quad 0 \le x \le 1 \\ &u = a_2\operatorname{sh}\omega(l-x), \quad \omega = \sqrt{q_0-\lambda}, \quad \omega\cos\nu\operatorname{ch}\omega a - \nu\sin\nu\operatorname{sh}\omega a = 0 \\ &u = a_2\sin\omega(l-x), \quad \omega = \sqrt{\lambda-q_0}, \quad \omega\cos\nu\cos\omega a - \nu\sin\nu\sin\omega a = 0 \\ &1 < x \le l, \quad \nu_n(0) = (n-1/2)\pi, \quad n = 1, 2, \dots \end{aligned} \tag{4.9}$$

According to relation (4.9) the natural frequencies $\omega_n(a) = \sqrt{\lambda_n(a)}$ decrease as the parameter $a \ge 0$ increases, as in the case when both ends are fixed (4.8). The qualitative conclusions reached above for the case when $q_0 = (n^*\pi)^2$ hold. Hence, the results of an analytical investigation of problem (4.1) clearly illustrate the conclusions of the theorem (see Section 2). They characterize the anomalous behaviour of the natural frequencies as the length of the elastic systems increase.

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